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The *n*-level spectral correlations for chaotic systems

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Abstract
We study the *n*-level spectral correlation functions of classically chaotic quantum systems without time-reversal symmetry. According to Bohigas, Giannoni and Schmit's universality conjecture, it is expected that the correlation functions are in agreement with the prediction of the circular unitary ensemble (CUE) of random matrices. A semiclassical resummation formalism allows us to express the correlation functions as sums over pseudo-orbits. Using an extended version of the diagonal approximation on the pseudo-orbit sums, we derive the *n*-level correlation functions identical to the $n \times n$ determinantal correlation functions of the CUE.

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1. Introduction

Quantum systems whose classical limit is chaotic display universal spectral statistics. Their spectral correlations depend only on the symmetry class of the system and agree with predictions obtained from averaging over ensembles of random matrices [1–4]. For instance, let us suppose that the time-reversal invariance of a chaotic system is broken by applying a magnetic field. Then, in the semiclassical limit, it is conjectured that its spectral correlation functions are in agreement with the predictions of the circular unitary ensemble (CUE) (or the Gaussian unitary ensemble (GUE)) of random matrices.

A way to understand the origins for this universality is provided by semiclassics. In the semiclassical theory, the 2-level spectral correlation function is expressed as a sum over the pairs of periodic orbits. Berry introduced a useful scheme called the diagonal approximation [5]. In this scheme, when the time-reversal invariance is broken, only the sum over the pairs of identical periodic orbits is taken into account. Then one can derive the smooth (non-oscillatory) part of the correlation function in agreement with the CUE. Shukla extended this scheme to calculate the *n*-level spectral correlation functions and succeeded in deriving asymptotic forms of the Fourier transforms in agreement with the CUE [6].

The diagonal approximation brought about great progress in understanding universality. However, it is able to only partially reproduce the random matrix predictions, even if we restrict ourselves to the case of broken time-reversal symmetry. It is an approximation which yields only the smooth parts of the correlation functions and misses the remaining oscillatory parts.

Refined schemes to reproduce the full predictions have been developed in the study of the Riemann zeta function. It is conjectured that the complex zeros of the Riemann zeta function are mutually correlated in a similar way to the energy levels of chaotic quantum systems without time-reversal symmetry [7–10]. Correlation functions of the zeros can be written as the multiple sums over prime numbers similar to the periodic orbit sums in semiclassics. One is thus able to develop a scheme analogous to the semiclassical periodic orbit theory. Using additional input from number theory, it is then possible to access oscillatory contributions as well. As a result, the full CUE correlation functions have been reproduced under certain assumptions in several works [11–15].

In the case of chaotic quantum systems, although the problem is in some respect more involved, the analogous question was addressed in [16–18]. Following progress in the method of semiclassical diagrammatic expansions [19–22], Heusler *et al* proposed a way to evaluate semiclassically both oscillatory and smooth parts of the full 2-level correlation function, and obtained results agreeing with the random matrix prediction [17]. Keating and Müller recently gave a justification of Heusler *et al*'s argument [18]. The essential idea is to relate the correlation function to a generating function (a ratio of spectral determinants) and then make use of an improved ('resummed') semiclassical approximation for the latter, the so-called Riemann–Siegel lookalike formula established by Berry and Keating [23–25]. For systems without time-reversal invariance, a generalization of the diagonal approximation to this setting is sufficient to derive the full 2-level correlation function.

In this paper, we apply a generalization of Keating and Müller's method to calculate the n-level correlation functions of chaotic quantum systems without time-reversal symmetry. In section 2, we develop a generating function formalism for the n-level correlation functions, introduce the Riemann–Siegel lookalike formula and put it into the formalism. In section 3, an extended version of the diagonal approximation is formulated and the n-level correlation functions are calculated in the semiclassical limit. The resulting formulas involve sums over several different contributions generalizing the sum over the smooth and oscillatory parts for the 2-level correlation function. In section 4, we verify that these sums are identical to the $n \times n$ determinantal formulas of the CUE correlation functions known from random-matrix theory (RMT). This is done by establishing their agreement with a representation of the random-matrix average obtained by Conrey and Snaith [15] (interestingly, this representation had originally been developed to facilitate the comparison to number-theoretic rather than semiclassical results). We thus confirm that the known partial results based on the diagonal approximation are extended to the forms in agreement with the full random matrix predictions. The last section is devoted to a brief discussion on the result.

2. Generating function

Let us suppose that H denotes the Hamiltonian of a bounded quantum system which is chaotic in the classical limit. We are interested in the distribution of the energy levels E_j (the eigenvalues of H). The density of these energy levels

$$\rho(E) = \sum_j \delta(E - E_j) \tag{2.1}$$

may be separated into the smoothed part

$$\bar{\rho}(E) \sim \frac{\Omega(E)}{(2\pi\hbar)^f} \tag{2.2}$$

and the fluctuation around it. Here $\Omega(E)$ is the volume of the energy shell in the classical phase space and $f > 1$ is the number of degrees of freedom.

We now want to determine the n-level correlation functions, which are defined as

$$R_n(\epsilon_1, \ldots, \epsilon_n) = \frac{1}{\bar{\rho}^n} \left\langle \prod_{j=1}^{n} \rho(E + \epsilon_j) \right\rangle \tag{2.3}$$

and describe the fluctuation of the energy level distribution around the smoothed density $\bar{\rho}$. To obtain a smooth function, we take the average $\langle\cdot\rangle$ over the windows of the center energy E and energy differences ϵ_j.

The idea of the generating function formalism is to represent the level densities in (2.3) through traces of the resolvent,

$$\rho(E) = \frac{\mathrm{i}}{2\pi} \left(\mathrm{Tr}\frac{1}{E^+ - H} - \mathrm{Tr}\frac{1}{E^- - H} \right) \tag{2.4}$$

(where $E^{\pm} = E \pm \mathrm{i}\kappa$, and κ is an infinitesimal positive number), and then express these traces in terms of derivatives of the spectral determinant $\Delta(E) = \det(E - H)$,

$$\mathrm{Tr}\frac{1}{E - H} = -\left.\frac{\partial}{\partial\epsilon}\frac{\Delta(E)}{\Delta(E+\epsilon)}\right|_{\epsilon=0}. \tag{2.5}$$

Equation (2.3) then turns into

$$\begin{aligned} R_n(\epsilon_1, \ldots, \epsilon_n) &= \left(\frac{\mathrm{i}}{2\pi\bar{\rho}}\right)^n \left\langle \prod_{j=1}^{n} \left\{ \sum_{\sigma_j=\pm 1} \sigma_j \mathrm{Tr}\frac{1}{E + \epsilon_j + \mathrm{i}\sigma_j\kappa_j - H} \right\} \right\rangle \\ &= \left.\frac{\partial^n}{\partial\epsilon_1\partial\epsilon_2\cdots\partial\epsilon_n} Z_n\right|_{\eta=\epsilon}, \end{aligned} \tag{2.6}$$

where

$$\boldsymbol{\epsilon} = (\epsilon_1, \epsilon_2, \ldots, \epsilon_n), \qquad \boldsymbol{\eta} = (\eta_1, \eta_2, \ldots, \eta_n), \tag{2.7}$$

and the generating function Z_n is defined as

$$Z_n = \left(\frac{1}{2\pi\bar{\rho}\mathrm{i}}\right)^n \left\langle \prod_{j=1}^{n} \left\{ \sum_{\sigma_j=\pm 1} \sigma_j \frac{\Delta(E + \eta_j)}{\Delta(E + \epsilon_j + \mathrm{i}\sigma_j\kappa_j)} \right\} \right\rangle, \kappa_j \downarrow 0. \tag{2.8}$$

We now derive a semiclassical approximation for Z_n. Using Gutzwiller's trace formula for chaotic systems [26], we can express the trace of the resolvent

$$g(E^+) = \mathrm{Tr}\frac{1}{E^+ - H} \tag{2.9}$$

as a sum over classical periodic orbits a,

$$g(E^+) = \bar{g}(E^+) - \frac{\mathrm{i}}{\hbar} \sum_a F_a T_a \,\mathrm{e}^{\mathrm{i}S_a(E^+)/\hbar}. \tag{2.10}$$

Here F_a is the stability amplitude (including the Maslov phase), S_a is the classical action and $T_a = \mathrm{d}S_a/\mathrm{d}E$ is the period of a. The smoothed part of the trace resolvent is written as $\bar{g}(E)$.

It follows from (2.10) that

$$\Delta(E^+) \propto \exp\left(\int^{E^+} g(E')\mathrm{d}E'\right)$$
$$\propto \exp\left(-\mathrm{i}\pi\bar{N}(E^+) - \sum_a F_a\,\mathrm{e}^{\mathrm{i}S_a(E^+)/\hbar}\right). \tag{2.11}$$

Here $\bar{N}(E)$ is the smoothed part of the cumulative energy-level density: it satisfies a relation $\bar{\rho} = \mathrm{d}\bar{N}/\mathrm{d}E$ with the smoothed part of the energy-level density.

Let us expand the exponential function and write (2.11) as a sum over pseudo-orbits A (a pseudo-orbit is a set of component periodic orbits):

$$\Delta(E^+) \propto \mathrm{e}^{-\mathrm{i}\pi\bar{N}(E^+)}\sum_A F_A(-1)^{n_A}\,\mathrm{e}^{\mathrm{i}S_A(E^+)/\hbar}. \tag{2.12}$$

Here n_A is the number of the component orbits a, S_A is the sum of S_a and F_A is the product of F_a. The factor F_A also includes the correction to the sign factor $(-1)^{n_A}$, when identical orbit copies are contained in A. We also find from (2.11) that the inverse of the spectral determinant is expanded as

$$\Delta(E^+)^{-1} \propto \mathrm{e}^{\mathrm{i}\pi\bar{N}(E^+)}\sum_A F_A\,\mathrm{e}^{\mathrm{i}S_A(E^+)/\hbar}. \tag{2.13}$$

Similar formulas for $\Delta(E^-)$ and $\Delta(E^-)^{-1}$ are obtained by complex conjugation.

However, these results do not yet incorporate the unitarity of the quantum-mechanical time evolution, and thus the fact that the energy levels are real. Berry and Keating argued that the unitarity requirement of the quantum dynamics leads to an approximation

$$\Delta(E) \propto \mathrm{e}^{-\mathrm{i}\pi\bar{N}(E)}\sum_{A(T_A<T_H/2)} F_A(-1)^{n_A}\,\mathrm{e}^{\mathrm{i}S_A(E)/\hbar} + \mathrm{c.c.} \tag{2.14}$$

for a real E. This formula is called the Riemann–Siegel lookalike formula [23–25] after a similar expression in the theory of the Riemann zeta function. Here the contributions of 'long' pseudo-orbits (pseudo-orbits for which the sum T_A of the periods of the component orbits is larger than half the Heisenberg time $T_H = 2\pi\hbar\bar{\rho}(E)$) in (2.12) are replaced by the complex conjugate of the contribution from the shorter pseudo-orbits.

Putting these results into (2.8), we obtain an expression

$$Z_n = \left(\frac{1}{2\pi\bar{\rho}\mathrm{i}}\right)^n \sum_{\substack{\sigma_j=\pm1\\ \tau_j=\pm1}} \left\langle \exp\left[\mathrm{i}\pi\sum_{j=1}^n\{\sigma_j\bar{N}(E+\epsilon_j) - \tau_j\bar{N}(E+\eta_j)\}\right]\right.$$
$$\left.\times\prod_{j=1}^n\left\{\sigma_j\sum_{A_j}F_{A_j}^{(\sigma_j)}\,\mathrm{e}^{\mathrm{i}\sigma_jS_{A_j}(E+\epsilon_j)/\hbar}\sum_{B_j(T_{B_j}<T_H/2)}F_{B_j}^{(\tau_j)}(-1)^{n_{B_j}}\,\mathrm{e}^{\mathrm{i}\tau_jS_{B_j}(E+\eta_j)/\hbar}\right\}\right\rangle \tag{2.15}$$

with $F_A^{(1)} = F_A$ and $F_A^{(-1)} = F_A^*$ (an asterisk means a complex conjugate). Here the sums over A_j originate from using (2.13) and its complex conjugate in the denominator of (2.8), whereas the sums over B_j result from applying (2.14) to the numerator. The sums over τ_j make sure that both summands in (2.14) are taken into account.

Most of the terms in the sum over σ_j and τ_j vanish when they are averaged over E, due to the highly oscillatory phase factor. Expanding the exponent of the phase factor involving

$\bar{N}$ in (2.15) as

$$\exp\left[\mathrm{i}\pi\sum_{j=1}^{n}\{\sigma_j\bar{N}(E+\epsilon_j)-\tau_j\bar{N}(E+\eta_j)\}\right]$$
$$\sim\exp\left[\mathrm{i}\pi\bar{N}(E)\sum_{j=1}^{n}(\sigma_j-\tau_j)+\mathrm{i}\pi\bar{\rho}(E)\sum_{j=1}^{n}(\sigma_j\epsilon_j-\tau_j\eta_j)\right], \tag{2.16}$$

we see that such cancellations can be avoided when

$$\sum_{j=1}^{n}(\sigma_j-\tau_j)=0 \tag{2.17}$$

holds. Hereafter we concentrate on the terms satisfying (2.17).

3. The *n*-level correlation functions in the semiclassical limit

To proceed, we introduce four sets I, J, K and L so that

$$\begin{aligned} I &= \{j|\sigma_j=1\},\\ J &= \{j|\sigma_j=-1\},\\ K &= \{j|\tau_j=1\},\\ L &= \{j|\tau_j=-1\}. \end{aligned} \tag{3.1}$$

It follows from (2.17) that

$$|I|=|K|, \tag{3.2}$$

where $|\mathcal{M}|$ is the number of the elements when $\mathcal{M}$ is a set. With this notation the phase factor involving the actions in (2.15) can be written as

$$\exp\left[\mathrm{i}\sum_{j=1}^{n}\{\sigma_jS_{A_j}(E+\epsilon_j)+\tau_jS_{B_j}(E+\eta_j)\}/\hbar\right]$$
$$=\exp\left[\mathrm{i}\left\{\sum_{j\in I}S_{A_j}(E+\epsilon_j)-\sum_{k\in J}S_{A_k}(E+\epsilon_k)\right.\right.$$
$$\left.\left.+\sum_{j\in K}S_{B_j}(E+\eta_j)-\sum_{k\in L}S_{B_k}(E+\eta_k)\right\}/\hbar\right]. \tag{3.3}$$

In the semiclassical limit, this phase factor also oscillates rapidly for most choices of pseudo-orbits, meaning that the corresponding summands in (2.15) will be averaged to zero. In order to find the dominant contribution, we need to choose the terms with nearly vanishing exponents. For that purpose, we simply assume that the component orbits in A_j, $j\in I$, and B_j, $j\in K$ (contributing with a positive sign in (3.3)), are the same as those in A_k, $k\in J$, and B_k, $k\in L$ (contributing with a negative sign), neglecting repetitions. We call this scheme the extended diagonal approximation, as it is a natural extension of Berry's diagonal approximation [5].

Within the diagonal approximation we can now drop the upper limits $T_H/2$ for the pseudo-orbits in (2.15). This is possible because each periodic orbit is now a common component of two pseudo-orbits. Hence its stability amplitude is coupled with the complex conjugate to form the absolute square in (2.15). Weighing orbits with the absolute square of their stability amplitude is sufficient to ensure convergence even without an upper limit on the sum of periods (for energies with an arbitrarily small imaginary part) [18].

In the extended diagonal approximation, each pseudo-orbit A_j $(j \in I)$ is a union of the disjoint sets $A_j \cap A_k$ $(k \in J)$ and $A_j \cap B_k$ $(k \in L)$. Consequently we find a decomposition

$$\sum_{A_j} F_{A_j}\, \mathrm{e}^{\mathrm{i}S_{A_j}(E+\epsilon_j)/\hbar} = \prod_{k\in J}\left(\sum_{A_j\cap A_k} F_{A_j\cap A_k}\, \mathrm{e}^{\mathrm{i}S_{A_j\cap A_k}(E+\epsilon_j)/\hbar}\right) \times \prod_{k\in L}\left(\sum_{A_j\cap B_k} F_{A_j\cap B_k}\, \mathrm{e}^{\mathrm{i}S_{A_j\cap B_k}(E+\epsilon_j)/\hbar}\right), \; j \in I. \tag{3.4}$$

Analogous decompositions apply to all other pseudo-orbit sums in (2.15). For instance, each pseudo-orbit A_j $(j \in J)$ is a union of the disjoint sets $A_j \cap A_k$ $(k \in I)$ and $A_j \cap B_k$ $(k \in K)$. It follows that

$$\sum_{A_j} F^*_{A_j}\, \mathrm{e}^{-\mathrm{i}S_{A_j}(E+\epsilon_j)/\hbar} = \prod_{k\in I}\left(\sum_{A_j\cap A_k} F^*_{A_j\cap A_k}\, \mathrm{e}^{-\mathrm{i}S_{A_j\cap A_k}(E+\epsilon_j)/\hbar}\right) \times \prod_{k\in K}\left(\sum_{A_j\cap B_k} F^*_{A_j\cap B_k}\, \mathrm{e}^{-\mathrm{i}S_{A_j\cap B_k}(E+\epsilon_j)/\hbar}\right), \; j \in J. \tag{3.5}$$

Similarly for $j \in K$ and $j \in L$ we obtain the decompositions

$$\begin{aligned}&\sum_{B_j} F_{B_j}(-1)^{n_{B_j}}\, \mathrm{e}^{\mathrm{i}S_{B_j}(E+\eta_j)/\hbar}\\ &= \prod_{k\in J}\left(\sum_{B_j\cap A_k} F_{B_j\cap A_k}(-1)^{n_{B_j\cap A_k}}\, \mathrm{e}^{\mathrm{i}S_{B_j\cap A_k}(E+\eta_j)/\hbar}\right)\\ &\times \prod_{k\in L}\left(\sum_{B_j\cap B_k} F_{B_j\cap B_k}(-1)^{n_{B_j\cap B_k}}\, \mathrm{e}^{\mathrm{i}S_{B_j\cap B_k}(E+\eta_j)/\hbar}\right), \; j \in K,\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}&\sum_{B_j} F^*_{B_j}(-1)^{n_{B_j}}\, \mathrm{e}^{-\mathrm{i}S_{B_j}(E+\eta_j)/\hbar}\\ &= \prod_{k\in I}\left(\sum_{B_j\cap A_k} F^*_{B_j\cap A_k}(-1)^{n_{B_j\cap A_k}}\, \mathrm{e}^{-\mathrm{i}S_{B_j\cap A_k}(E+\eta_j)/\hbar}\right)\\ &\times \prod_{k\in K}\left(\sum_{B_j\cap B_k} F^*_{B_j\cap B_k}(-1)^{n_{B_j\cap B_k}}\, \mathrm{e}^{-\mathrm{i}S_{B_j\cap B_k}(E+\eta_j)/\hbar}\right), \; j \in L.\end{aligned} \tag{3.7}$$

If we substitute (3.4)–(3.7) into (2.15), we see that as anticipated the actions almost compensate. The only remaining action difference is due to the energy arguments being slightly different. This difference can be approximated using expansions of the type

$$S_A(E+\epsilon_j) - S_A(E+\epsilon_k) \sim T_A(\epsilon_j - \epsilon_k). \tag{3.8}$$

Our result thus boils down to

$$Z_n = \left(\frac{1}{2\pi\bar{\rho}\mathrm{i}}\right)^n \sum_{\substack{\sigma_j,\tau_j \\ (\sum_{j=1}^n \sigma_j = \sum_{j=1}^n \tau_j)}} \prod_{j=1}^n \sigma_j \,\mathrm{e}^{\mathrm{i}\pi\bar{\rho}(E)(\sigma_j\epsilon_j - \tau_j\eta_j)}$$
$$\times \left\langle \frac{\prod_{j\in I,k\in L} \zeta(-\mathrm{i}(\epsilon_j-\eta_k)/\hbar) \prod_{j\in K,k\in J} \zeta(-\mathrm{i}(\eta_j-\epsilon_k)/\hbar)}{\prod_{j\in I,k\in J} \zeta(-\mathrm{i}(\epsilon_j-\epsilon_k)/\hbar) \prod_{j\in K,k\in L} \zeta(-\mathrm{i}(\eta_j-\eta_k)/\hbar)} \right\rangle,$$

where the sums over intersections of pseudo-orbits (with the remaining action differences (3.8)) were written in terms of the dynamical zeta function:

$$\zeta(s) = \sum_A |F_A|^2 (-1)^{n_A} \,\mathrm{e}^{-sT_A},$$
$$\zeta(s)^{-1} = \sum_A |F_A|^2 \,\mathrm{e}^{-sT_A}. \tag{3.9}$$

In order to see the asymptotic behavior of Z_n in the semiclassical limit, we introduce the rescaling

$$\epsilon_j \mapsto \frac{\epsilon_j}{2\pi\bar{\rho}}, \qquad j = 1, 2, \ldots, n \tag{3.10}$$

and, noting (2.2), utilize the asymptotic formula

$$\zeta(s) \propto s, \qquad s \to 0, \tag{3.11}$$

which holds for chaotic systems [27]. In the semiclassical limit $\hbar \to 0$, we now obtain

$$Z_n = \left(\frac{1}{2\pi\bar{\rho}\mathrm{i}}\right)^n \sum_{\substack{\sigma_j,\tau_j \\ (\sum_{j=1}^n \sigma_j = \sum_{j=1}^n \tau_j)}} (-1)^{|L|} \,\mathrm{e}^{\mathrm{i}(\sum_{j\in I}\epsilon_j - \sum_{j\in J}\epsilon_j - \sum_{j\in K}\eta_j + \sum_{j\in L}\eta_j)/2}$$
$$\times \frac{\prod_{\substack{j\in I\\k\in L}} (\epsilon_j - \eta_k) \prod_{\substack{j\in K\\k\in J}} (\eta_j - \epsilon_k)}{\prod_{\substack{j\in I\\k\in J}} (\epsilon_j - \epsilon_k) \prod_{\substack{j\in K\\k\in L}} (\eta_j - \eta_k)}. \tag{3.12}$$

To compare this result to the random-matrix expression in [15], it is helpful to adopt a slightly different notation. The sum over all choices for the sign factors $\tau_j = \pm 1$, $j = 1, 2, \ldots, n$, is equivalent to summation over all ways to write the set $\{1, 2, \ldots, n\}$ as a direct sum of two subsets K and L. The direct sum of disjoint sets K and L is defined as the union $K \cup L$ and denoted by $K + L$. The corresponding arguments $\mathrm{i}\epsilon_j$, $\mathrm{i}\eta_j$ in (3.12) then form sets

$$\begin{aligned} A &= \{\mathrm{i}\epsilon_j | j \in K\}, \qquad & B &= \{-\mathrm{i}\epsilon_j | j \in L\}, \\ C &= \{\mathrm{i}\eta_j | j \in K\}, \qquad & D &= \{-\mathrm{i}\eta_j | j \in L\}. \end{aligned} \tag{3.13}$$

Moreover the sum over signs $\sigma_j = \pm 1$ determining the sets I and J can be replaced by a sum over subsets

$$\begin{aligned} S &= \{\mathrm{i}\epsilon_j | j \in J \cap K\} \subset A, \\ T &= \{-\mathrm{i}\epsilon_j | j \in I \cap L\} \subset B. \end{aligned} \tag{3.14}$$

Indeed, if we define

$$\bar{S} = A - S, \qquad \bar{T} = B - T, \tag{3.15}$$

and $\mathcal{M}^- = \{-\alpha|\alpha \in \mathcal{M}\}$ when $\mathcal{M}$ is a set, the sets of energy increments corresponding to I and J can be expressed through S and T as

$$\begin{aligned} \bar{S} + T^- &= \{\mathrm{i}\epsilon_j | j \in I\}, \\ \bar{T} + S^- &= \{-\mathrm{i}\epsilon_j | j \in J\}. \end{aligned} \tag{3.16}$$

With these definitions, when we apply the rescaling (3.10) to (2.6), equation (3.12) yields

$$R_n(\epsilon_1, \ldots, \epsilon_n) = \sum_{K+L=\{1,2,\ldots,n\}} q(A; B), \tag{3.17}$$

where

$$q(A; B) = \prod_{\alpha\in A,\beta\in B} \frac{\partial}{\partial\alpha}\frac{\partial}{\partial\beta} r(A, B; C, D)\bigg|_{\eta=\epsilon} \tag{3.18}$$

and $r(A, B; C, D)$ is defined as

$$\begin{aligned} r(A, B; C, D) = \sum_{\substack{S\subset A, T\subset B\\(|S|=|T|)}} \exp\left\{\frac{1}{2}\left(\sum_{\alpha\in\bar{S}+T^-}\alpha + \sum_{\beta\in\bar{T}+S^-}\beta - \sum_{\gamma\in C}\gamma - \sum_{\delta\in D}\delta\right)\right\} \\ \times\ z(\bar{S} + T^-, \bar{T} + S^-; C, D) \end{aligned} \tag{3.19}$$

with

$$z(\mathcal{W}, \mathcal{X}; \mathcal{Y}, \mathcal{Z}) = \frac{\prod_{\substack{\alpha\in\mathcal{W}\\ \delta\in\mathcal{Z}}}(\alpha+\delta) \prod_{\substack{\beta\in\mathcal{X}\\ \gamma\in\mathcal{Y}}}(\beta+\gamma)}{\prod_{\substack{\alpha\in\mathcal{W}\\ \beta\in\mathcal{X}}}(\alpha+\beta) \prod_{\substack{\gamma\in\mathcal{Y}\\ \delta\in\mathcal{Z}}}(\gamma+\delta)}. \tag{3.20}$$

For example, the 2-level correlation function is calculated as

$$\begin{aligned} R_2(\epsilon_1, \epsilon_2) = q(\emptyset; \{-\mathrm{i}\epsilon_1, -\mathrm{i}\epsilon_2\}) + q(\{\mathrm{i}\epsilon_1\}; \{-\mathrm{i}\epsilon_2\}) \\ + q(\{\mathrm{i}\epsilon_2\}; \{-\mathrm{i}\epsilon_1\}) + q(\{\mathrm{i}\epsilon_1, \mathrm{i}\epsilon_2\}; \emptyset), \end{aligned} \tag{3.21}$$

where

$$\begin{aligned} q(\emptyset; \{-\mathrm{i}\epsilon_1, -\mathrm{i}\epsilon_2\}) &= -\frac{\partial^2}{\partial\epsilon_1\partial\epsilon_2} r(\emptyset, \{-\mathrm{i}\epsilon_1, -\mathrm{i}\epsilon_2\}; \emptyset, \{-\mathrm{i}\eta_1, -\mathrm{i}\eta_2\}) \\ &= -\frac{\partial^2}{\partial\epsilon_1\partial\epsilon_2}\, \mathrm{e}^{-\mathrm{i}(\epsilon_1+\epsilon_2-\eta_1-\eta_2)/2}\bigg|_{\eta=\epsilon} \\ &= \frac{1}{4}, \end{aligned} \tag{3.22}$$

$$\begin{aligned} q(\{\mathrm{i}\epsilon_1\}; \{-\mathrm{i}\epsilon_2\}) &= \frac{\partial^2}{\partial\epsilon_1\partial\epsilon_2} r(\{\mathrm{i}\epsilon_1\}, \{-\mathrm{i}\epsilon_2\}; \{\mathrm{i}\eta_1\}, \{-\mathrm{i}\eta_2\}) \\ &= \frac{\partial^2}{\partial\epsilon_1\partial\epsilon_2}\left\{\mathrm{e}^{\mathrm{i}(\epsilon_1-\epsilon_2-\eta_1+\eta_2)/2}\frac{(\epsilon_1-\eta_2)(\eta_1-\epsilon_2)}{(\epsilon_1-\epsilon_2)(\eta_1-\eta_2)}\right. \\ &\quad \left.+\ \mathrm{e}^{\mathrm{i}(\epsilon_2-\epsilon_1-\eta_1+\eta_2)/2}\frac{(\epsilon_2-\eta_2)(\eta_1-\epsilon_1)}{(\epsilon_2-\epsilon_1)(\eta_1-\eta_2)}\right\}\bigg|_{\eta=\epsilon} \\ &= \frac{1}{4} - \frac{1-\mathrm{e}^{-\mathrm{i}(\epsilon_1-\epsilon_2)}}{(\epsilon_1-\epsilon_2)^2}, \end{aligned} \tag{3.23}$$

$$q(\{\mathrm{i}\epsilon_2\};\{-\mathrm{i}\epsilon_1\})=\frac{1}{4}-\frac{1-\mathrm{e}^{\mathrm{i}(\epsilon_1-\epsilon_2)}}{(\epsilon_1-\epsilon_2)^2},\tag{3.24}$$

$$q(\{\mathrm{i}\epsilon_1,\mathrm{i}\epsilon_2\};\emptyset)=\frac{1}{4}.\tag{3.25}$$

Putting (3.22)–(3.25) into (3.21), we can readily find a compact expression

$$R_2(\epsilon_1,\epsilon_2)=1-\left[\frac{\sin\{(\epsilon_1-\epsilon_2)/2\}}{(\epsilon_1-\epsilon_2)/2}\right]^2.\tag{3.26}$$

4. Determinant expressions

In [15], Conrey and Snaith analyzed the circular unitary ensemble (CUE) of random matrices. Based on the ratios theorem [28, 29] (see also [30, 31]) on the characteristic polynomials, they established the formulas

$$R_n^{(\mathrm{CUE})}(\epsilon_1,\ldots,\epsilon_n)=\sum_{K+L+M=1,2,\ldots,n}\tilde{q}(A;B)\tag{4.1}$$

for the scaled n-eigenparameter correlation functions $R_n^{(\mathrm{CUE})}$. Here the union of the three disjoint sets K, L and M is $\{1,2,\ldots,n\}$ and

$$\tilde{q}(A;B)=\prod_{\alpha\in A,\beta\in B}\frac{\partial}{\partial\alpha}\frac{\partial}{\partial\beta}\tilde{r}(A,B;C,D)\bigg|_{\eta=\epsilon}\tag{4.2}$$

with

$$\tilde{r}(A,B;C,D)=\sum_{\substack{S\subset A,T\subset B\\(|S|=|T|)}}\exp\left\{-\sum_{\alpha\in S}\alpha-\sum_{\beta\in T}\beta\right\}z(\bar{S}+T^-,\bar{T}+S^-;C,D).\tag{4.3}$$

The definitions of A, B, C, D, $\bar{S}$, $\bar{T}$ and $z(\mathcal{W},\mathcal{X};\mathcal{Y},\mathcal{Z})$ are the same as before ((3.13), (3.15) and (3.20)).

On the other hand, it is well known that the scaled n-eigenparameter correlation functions of the CUE have determinant expressions [32]:

$$R_n^{(\mathrm{CUE})}(\epsilon_1,\ldots,\epsilon_n)=\det\left[\frac{\sin\{(\epsilon_j-\epsilon_k)/2\}}{(\epsilon_j-\epsilon_k)/2}\right]_{j,k=1,2,\ldots,n}.\tag{4.4}$$

Therefore, in order to verify the same determinant expressions for the scaled semiclassical n-level correlation functions (3.17), it is sufficient to prove that (3.17) is identical to (4.1).

To do this, we note that the derivatives $\frac{\partial}{\partial\alpha}$ $(\alpha\in A)$, $\frac{\partial}{\partial\beta}$ $(\beta\in B)$ in (3.18) may act either on the phase factor

$$\phi=\exp\left\{\frac{1}{2}\left(\sum_{\alpha\in\bar{S}+T^-}\alpha+\sum_{\beta\in\bar{T}+S^-}\beta-\sum_{\gamma\in C}\gamma-\sum_{\delta\in D}\delta\right)\right\}\tag{4.5}$$

or on $z(\bar{S}+T^-,\bar{T}+S^-;C,D)$. We can thus sum over all ways to split A into two disjoint subsets A_1 and A_2, and then let the derivatives with respect to the elements of A_1 act on ϕ whereas the derivatives with respect to the elements of A_2 act on z. The corresponding sets of indices in K are denoted by K_1 and K_2. Analogously B is divided into subsets B_1 (with derivatives acting on ϕ) and B_2 (with derivatives acting on z), and the corresponding sets of indices in L are denoted by L_1 and L_2. This yields

$$\begin{aligned} R_n(\epsilon_1,\ldots,\epsilon_n) &= \sum_{K+L=\{1,2,\ldots,n\}} \sum_{\substack{S\subset A, T\subset B(|S|=|T|)\\ A_1+A_2=A, B_1+B_2=B}} \left(\prod_{\alpha\in A_1,\beta\in B_1} \frac{\partial}{\partial\alpha}\frac{\partial}{\partial\beta}\phi\right) \\ &\times \left\{ \prod_{\alpha\in A_2,\beta\in B_2} \frac{\partial}{\partial\alpha}\frac{\partial}{\partial\beta} z(\bar{S}+T^-,\bar{T}+S^-;C,D)\right\}\Bigg|_{\eta=\epsilon}. \end{aligned} \tag{4.6}$$

Now it is important that the derivatives

$$\prod_{\alpha\in A_2,\beta\in B_2} \frac{\partial}{\partial\alpha}\frac{\partial}{\partial\beta} z(\bar{S}+T^-,\bar{T}+S^-;C,D)\Bigg|_{\eta=\epsilon} \tag{4.7}$$

in (4.6) are nonzero only if

$$S\subset A_2 \qquad \text{and} \qquad T\subset B_2 \tag{4.8}$$

hold. This is because, for each element $\mathrm{i}\epsilon_j\in S$, $-\mathrm{i}\epsilon_j$ is included in $S^-\subset \bar{T}+S^-$ and the corresponding element $\mathrm{i}\eta_j$ is included in C so that $z(\bar{S}+T^-,\bar{T}+S^-;C,D)$ defined in (3.20) contains a factor $-\mathrm{i}\epsilon_j+\mathrm{i}\eta_j$ vanishing for $\boldsymbol{\eta}=\boldsymbol{\epsilon}$. Nonzero contributions arise only if all such terms are eliminated by differentiating z with respect to all elements $\mathrm{i}\epsilon_j\in S$. We thus need to have $S\subset A_2$. Analogous reasoning leads to $T\subset B_2$.

Now let us consider the phase factor ϕ. Each derivative of ϕ with respect to the elements $\alpha\in A_1\subset\bar{S}$ and $\beta\in B_1\subset\bar{T}$ leads to a factor $\frac{1}{2}$. If we subsequently identify $\boldsymbol{\eta}=\boldsymbol{\epsilon}$, the exponent of ϕ turns into

$$\frac{1}{2}\left(\sum_{\alpha\in\bar{S}+T^-}\alpha+\sum_{\beta\in\bar{T}+S^-}\beta-\sum_{\gamma\in C=A=S+\bar{S}}\gamma-\sum_{\delta\in D=B=T+\bar{T}}\delta\right) = -\sum_{\alpha\in S}\alpha-\sum_{\beta\in T}\beta. \tag{4.9}$$

We thus obtain

$$\begin{aligned} R_n(\epsilon_1,\ldots,\epsilon_n) = &\sum_{K+L=\{1,2,\ldots,n\}} \sum_{\substack{S\subset A_2\subset A, T\subset B_2\subset B\\ (|S|=|T|)}} \frac{1}{2^{|A_1|+|B_1|}} \\ &\times \mathrm{e}^{-\sum_{\alpha\in S}\alpha-\sum_{\beta\in T}\beta} \prod_{\alpha\in A_2,\beta\in B_2} \frac{\partial}{\partial\alpha}\frac{\partial}{\partial\beta} z(\bar{S}+T^-,\bar{T}+S^-;C,D)\Bigg|_{\eta=\epsilon}. \end{aligned} \tag{4.10}$$

Equation (4.10) can be further simplified if we use the fact that the summands in (4.10) do not depend on how the elements of A_1+B_1 are distributed among A_1 and B_1. In particular, we have

$$\begin{aligned} &\prod_{\alpha\in A_2,\beta\in B_2} \frac{\partial}{\partial\alpha}\frac{\partial}{\partial\beta} z(\bar{S}+T^-,\bar{T}+S^-;C,D)\Bigg|_{\eta=\epsilon} \\ &= \prod_{\alpha\in A_2,\beta\in B_2} \frac{\partial}{\partial\alpha}\frac{\partial}{\partial\beta} z(\bar{S}\cap A_2+T^-,\bar{T}\cap B_2+S^-;C_2,D_2)\Bigg|_{\eta=\epsilon}, \end{aligned} \tag{4.11}$$

where

$$\begin{aligned} C_2 &= \{\mathrm{i}\eta_j \,|\, j\in K_2\}\subset C, \\ D_2 &= \{-\mathrm{i}\eta_j \,|\, j\in L_2\}\subset D \end{aligned} \tag{4.12}$$

and all sets on the rhs of (4.11) can be shown to exclude A_1 and B_1.

Let us prove (4.11). It follows from (4.8) that $\bar{S} = \bar{S} \cap A_1 + \bar{S} \cap A_2 = A_1 + \bar{S} \cap A_2$ and $\bar{T} = \bar{T} \cap B_1 + \bar{T} \cap B_2 = B_1 + \bar{T} \cap B_2$. Then, using $C_1 = C - C_2$ and $D_1 = D - D_2$, we obtain

$$\begin{aligned} z(\bar{S} + T^-, \bar{T} + S^-; C, D) &= \frac{\prod\limits_{\substack{\alpha \in A_1 + \bar{S} \cap A_2 + T^- \\ \delta \in D_1 + D_2}} (\alpha + \delta) \prod\limits_{\substack{\beta \in B_1 + \bar{T} \cap B_2 + S^- \\ \gamma \in C_1 + C_2}} (\beta + \gamma)}{\prod\limits_{\substack{\alpha \in A_1 + \bar{S} \cap A_2 + T^- \\ \beta \in B_1 + \bar{T} \cap B_2 + S^-}} (\alpha + \beta) \prod\limits_{\substack{\gamma \in C_1 + C_2 \\ \delta \in D_1 + D_2}} (\gamma + \delta)} \\ &= W_1 W_2 z(\bar{S} \cap A_2 + T^-, \bar{T} \cap B_2 + S^-; C_2, D_2), \end{aligned} \tag{4.13}$$

where

$$W_1 = \frac{\prod\limits_{\substack{\alpha \in A_1 \\ \delta \in D}} (\alpha + \delta) \prod\limits_{\substack{\beta \in B_1 \\ \gamma \in C}} (\beta + \gamma)}{\prod\limits_{\substack{\alpha \in A_1 \\ \beta \in B_1}} (\alpha + \beta) \prod\limits_{\substack{\gamma \in C_1 \\ \delta \in D_1}} (\gamma + \delta) \prod\limits_{\substack{\gamma \in C_1 \\ \delta \in D_2}} (\gamma + \delta) \prod\limits_{\substack{\gamma \in C_2 \\ \delta \in D_1}} (\gamma + \delta)} \tag{4.14}$$

and

$$\begin{aligned} W_2 &= \frac{\prod\limits_{\substack{\alpha \in \bar{S} \cap A_2 + T^- \\ \delta \in D_1}} (\alpha + \delta) \prod\limits_{\substack{\beta \in \bar{T} \cap B_2 + S^- \\ \gamma \in C_1}} (\beta + \gamma)}{\prod\limits_{\substack{\alpha \in A_1 \\ \beta \in \bar{T} \cap B_2 + S^-}} (\alpha + \beta) \prod\limits_{\substack{\alpha \in \bar{S} \cap A_2 + T^- \\ \beta \in B_1}} (\alpha + \beta)} \\ &= \prod_{\alpha \in \bar{S} \cap A_2 + T^-} \prod_{j \in L_1} \left(1 + \mathrm{i} \frac{\epsilon_j - \eta_j}{\alpha - \mathrm{i}\epsilon_j} \right) \prod_{\beta \in \bar{T} \cap B_2 + S^-} \prod_{j \in K_1} \left(1 + \mathrm{i} \frac{\eta_j - \epsilon_j}{\beta + \mathrm{i}\epsilon_j} \right). \end{aligned} \tag{4.15}$$

We can see that W_1 is independent of the elements of $A_2 \cup B_2$ and

$$\left. \frac{\partial}{\partial \alpha} W_2 \right|_{\eta = \epsilon} = 0 \tag{4.16}$$

for $\alpha \in A_2 \cup B_2$. (To check (4.16), note that only α and β in the last line of (4.15) may belong to A_2 or B_2. If we take derivatives with respect to any of these variables, one factor in the product turns into $-\mathrm{i}\frac{\epsilon_j - \eta_j}{(\alpha - \mathrm{i}\epsilon_j)^2}$ or $-\mathrm{i}\frac{\eta_j - \epsilon_j}{(\beta + \mathrm{i}\epsilon_j)^2}$ and vanishes after setting $\boldsymbol{\eta} = \boldsymbol{\epsilon}$.) Therefore nonvanishing contributions arise only if the derivatives of (4.13) with respect to the elements of $A_2 \cup B_2$ act on $z(\bar{S} \cap A_2 + T^-, \bar{T} \cap B_2 + S^-; C_2, D_2)$. Then W_1 and W_2 can be replaced by their special values

$$W_1|_{\eta = \epsilon} = W_2|_{\eta = \epsilon} = 1 \tag{4.17}$$

at $\boldsymbol{\eta} = \boldsymbol{\epsilon}$.

Thus equation (4.11) is proven, and it indeed becomes irrelevant how the elements of $A_1 + B_1$ are distributed among the two subsets, or how the corresponding indices in $M \equiv K_1 + L_1$ are distributed among K_1 and L_1. We can thus stop to discriminate between K_1 and L_1. This means that we trade the sums over K, L and over A_2, B_2 in (4.10) for one over $K_2 + L_2 + M = \{1, 2, \ldots, n\}$. Since there are $2^{|M|}$ ways to divide M into K_1 and L_1 we then need to multiply with $2^{|M|}$ which cancels the factor $\frac{1}{2^{|A_1|+|B_1|}} = \frac{1}{2^{|M|}}$.

The n-level correlation functions can thus be written as

$$R_n(\epsilon_1,\ldots,\epsilon_n) = \sum_{K_2+L_2+M=\{1,2,\ldots,n\}} \sum_{\substack{S\subset A_2, T\subset B_2\\(|S|=|T|)}} \mathrm{e}^{-\sum_{\alpha\in S}\alpha-\sum_{\beta\in T}\beta}$$

$$\times \prod_{\alpha\in A_2,\beta\in B_2} \frac{\partial}{\partial\alpha}\frac{\partial}{\partial\beta} z(\bar{S}\cap A_2+T^-, \bar{T}\cap B_2+S^-; C_2, D_2)\bigg|_{\eta=\epsilon}, \tag{4.18}$$

which is identical to the CUE n-eigenparameter correlation functions (4.1). The sets K_2, L_2, A_2, B_2 correspond to K, L, A, B in (4.1). The determinant expressions

$$R_n(\epsilon_1,\ldots,\epsilon_n) = \det\left[\frac{\sin\{(\epsilon_j-\epsilon_k)/2\}}{(\epsilon_j-\epsilon_k)/2}\right]_{j,k=1,2,\ldots,n} \tag{4.19}$$

for the n-level correlation functions are thus verified.

5. Discussion

In this paper the spectral correlation functions of chaotic quantum systems with broken time-reversal symmetry were investigated. Using the Riemann–Siegel lookalike formula and an extended version of Berry's diagonal approximation, we semiclassically evaluated the n-level correlation functions identical to the $n \times n$ determinantal predictions of random matrix theory.

Although the semiclassical results are exactly in agreement with the random matrix predictions, we admit that there are higher order nonzero terms in the semiclassical diagrammatic expansion, which are neglected in the extended diagonal approximation. The exact agreement suggests that those terms mutually cancel each other. Such a cancellation was diagrammatically verified in [17, 20–22] for the 2-level correlation function. It should also be verified for the general n-level correlation functions in future works.

As noted in the introduction, if the quantum system is symmetric under time-reversal, it belongs to a different universality class. In this case, the semiclassical argument becomes more difficult, because the higher order terms in the diagrammatic expansion are more involved and give a net contribution [17, 20–22]. The random matrix predictions are derived from the corresponding circular orthogonal ensemble (COE) (or Gaussian orthogonal ensemble (GOE)), and the n-level correlation functions have $2n \times 2n$ Pfaffian forms [33]. It would be interesting if difficulties are overcome and one is able to see how Pfaffian forms semiclassically appear.

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